# The dynamic differential forms of the Klein-Gordon field and the conformal group

RICHARD ARENS

Department of Mathematics
University of California, Los Angeles

**Abstract.** *Dynamic differential forms are the natural generalization of conserved currents. We discover the entire class for the real Klein-Gordon field, and find that each dynamic form is equivalent to a Noetherian form, that is, a form based on a canonical symmetry. Going to the complex case, we classify also its dynamic currents. It appears that some of them are definitely not equivalent to Noetherian forms.*

## 1. INTRODUCTION

An infinitesimal symmetry $U$ of the action integral of a field theory gives rise, via E. Noether's theorem, to a differential form $N(U)$ of degree 3 which is *dynamic*, i.e. is locally exact on each extremal for the action.

We present here a catalog of the dynamic currents for the Klein-Gordon field. The infinitesimal generators of the group of conformal transformations in Minkowski space occupy a prominent place in the description of these dynamic currents.

The main purpose of this catalog is to settle the question of whether each dynamic form $\delta$ coincides with or is at least equivalent in a natural way to a Noetherian form $N(U)$.

For the real 1-dimensional Klein-Gordon field, our catalog enables us to show that for each dynamic form there is a Noetherian form $N(U)$ equivalent to $\delta$ in a natural way (see below). The infinitesimal transformation (vector field) $U$ involved preserves the (generalized) symplectic structure of a space $\mathbb{R}^9$ of 9 dimensions, with coordinates $t^1, t^2, t^3, t^4$ for space-time, $x$ for the value of the field $\varphi$, and $p_1, \ldots, p_4$ for the values of the derivatives $\partial\varphi/\partial t^1, \ldots, \partial\varphi/\partial t^4$. The space-

-time component of this vector field $U$ has to be an infinitesimal *conformal* transformation.

It can be genuinely conformal for some non-linear variants of the Klein-Gordon field, such as $\varphi^4$, or, returning to the Klein-Gordon field proper, when $m = 0$; but when $m \neq 0$, then this space-time component can be at most an infinitesimal inhomogeneous Lorentz (i.e. Poincaré) transformation.

Moreover, each such dynamic form $\delta$ is naturally equivalent to a current, where this term means a 3-form of this sort:

$$\delta = J^1 dt^2 \wedge dt^3 \wedge dt^4 - J^2 dt^1 \wedge dt^3 \wedge dt^4 + \ldots - J^4 dt^1 \wedge dt^2 \wedge dt^3,$$

having (evidently) just four components which are functions of the $t^i, x$, and the $p_i$.

There are two vector fields one can make using these four components of $\delta$:

$$J^{(t)} = J^1 \frac{\partial}{\partial t^1} + \ldots + J^4 \frac{\partial}{\partial t^4}$$

and

$$J^{(p)} = J^1 \frac{\partial}{\partial p^1} + \ldots + J^4 \frac{\partial}{\partial p^4},$$

where $p^i = g^{ij} p_j$. Let $\varphi$ be a solution of $K. - G$. Replace $x$ in $J^{(t)}$ by $\varphi$, and the $p_i$ by $\partial\varphi / \partial t^i$. The exactness-on-extremals of $\delta$ just says that then the divergence of $J^{(t)}$ will vanish.

For $J^{(p)}$ a consequence of $\delta$ being dynamic is that $J^{(p)}$ is an infinitesimal conformal transformation relative to $p_1, \ldots, p_4$. This property is expressed mathematically by the system

$$\frac{\partial J_i}{\partial p^j} + \frac{\partial J_j}{\partial p^i} = 2\mu g_{ij} \qquad (i, j = 1, \ldots, 4),$$

where $\mu$ is a modulus of conformality. The general solution here is

$$J_i = \frac{1}{2} A_i p^j p_j - A_j p^j p_i + m_{ij} p^j + u p_i + e_i$$

and

$$\mu = u - A_i p^i$$

where $A_1, \ldots, A_4$, $m_{ij} (= -m_{ji})$, $u, e_1, \ldots, e_4$ depend only on $x$ and the $t$'s. Indeed, $A^1, \ldots, A^4$ depend only on the $t$'s, and

$$A^1 \frac{\partial}{\partial t^1} + \ldots + A^4 \frac{\partial}{\partial t^4} \equiv A$$

is the space-time component of the $U$ mentioned earlier. The $u$ satisfies the field equation. (The $m_{ij}$ and $e_j$ satisfy some other differential equations). For later reference, we call $A$ the *derived* field of $J^{(p)}$.

Most dynamic forms are not currents, but for each dynamic form $\delta$, there is a current $\epsilon$ such that the difference $\delta - \epsilon$ is a sum $\zeta + \eta$ where $\zeta$ vanishes on all extremals, and $\eta$ is exact. Thus $\delta$ is equivalent, in this natural sense, to a current.

We then show that a dynamic current $\epsilon$ is equivalent to a Noetherian form. This fact does not extend to the 2-dimensional, or complex, Klein-Gordon field, where the facts are as follows.

Let $\varphi$, $\psi$ be the field components. Let $p_i = \partial\varphi/\partial t^i$, $q_i = \partial\psi/\partial t^i$ $(i = 1, \ldots, 4)$. We find that for a dynamic current in this case,

$$J_i = J_i^{(1)} + J_i^{(2)} + I_i$$

where $J_i^{(1)}$ is a conformal field with derived field $A$, $J_i^{(2)}$ is a conformal field with derived field $B$. $J^{(1)}$ is conformal in terms of $p_1, \ldots, p_4$ and $J^{(2)}$ is conformal in terms of $q_1, \ldots, q_4$. The term $I_i$ is bilinear in the $p$ and $q$. (We give the details below). The derived fields $A$ and $B$ are infinitesimal conformal transformations. They may be different. In fact if $A \neq B$ then the dynamic current is not equivalent (in the natural sense) to any Noetherian form.

## 2. MAIN THEOREM ON DYNAMIC CURRENTS

We take $\mathbb{R}^4$ as our model for space-time and use $t^1, \ldots, t^4$ as the coordinates. Unconventionally, but to keep in evidence the relation to particle mechanics we use $x^1, \ldots, x^n$ as coordinates in the space $Q$ in which the field has its values. Therefore $t^1, \ldots, t^4, x^1, \ldots, x^n$ are coordinates for $\mathbb{R}^4 \times Q$.

The manifold $\Phi$ in which the dynamical forms, the action form $\alpha, \ldots$ are defined is the jet bundle $J^1(\mathbb{R}^4, Q)$ [5]. A jet can be regarded as a linear map $j$ from some tangent vector space $T^1(\mathbb{R}^4; m)$ in $\mathbb{R}^4$ to one, $T^1(Q, q)$ in $Q$; $t^i(j) = t^i(m)$ and $x^k(j) = x^k(q)$. There are more coordinates:

> $(x^k)_i(j)$ is the $k$-component of the image under $j$ of the $i$-th unit vector $\partial/\partial t^i$ [cfr. 1, 8.1], $k = 1, \ldots, n$.

When $\mathbb{R}^4$ is replaced by $\mathbb{R}^1$ then $(x^k)_1$ is the familiar $\dot{x}^k$.

A 3-form $\delta$ in our $\Phi$ will be called a current, or of current type, if

$$\delta = J^1 \mathrm{d}^{234} - J^2 \mathrm{d}^{134} + J^3 \mathrm{d}^{124} - J^4 \mathrm{d}^{123}, \tag{2.1}$$

where $d^{234}$ means $dt^2 \wedge dt^3 \wedge dt^4$, etc. We will omit the wedges in all cases.

We will often write $\delta = J^1 d^{234} - \ldots$, meaning (2.1).

In the real $n$-dimensional Klein-Gordon field, $Q = \mathbb{R}^n$; and the extremals are the solutions of

$$(2.2) \qquad -g^{ij} \frac{\partial^2 x^k}{\partial t^i \, \partial t^j} = m^2 x^k, \qquad k = 1, 2, \ldots, n.$$

A Lagrangian density $L$ for which (2.2) are the Euler equations is

$$(2.3) \qquad L = \frac{1}{2} \sum_{k=1}^{n} [g^{ij}(x^k)_i (x^k)_j - m^2 \{x^k\}^2].$$

*Caution*: Here $\{x^k\}^2$ is $x^k$ squared, and *not* perchance $(x^k)_i$ with its suffix $i$ raised by the use of $g^{i2}$. We will do that index raising very often, however.

We will eventually present a 4-form $\alpha$ (in Section 8 below) having the same extremals as (2.3). We go part of the way now in order to introduce some further notation.

We reduce the degree of this Lagrangean $L$ by introducing new variables $p_k^i$ and $x_i^k$, arriving at a new Lagrangean

$$L^* = \frac{1}{2} \sum_k [g^{ij} x_i^k x_j^k - m^2 x^k x^k] + \sum_k p_k^i [(x^k)_i - x_u^k]$$

which is linear in the derivatives $(x^k)_i$ and has the same extremals [1, 3.4]. We take the further step [1, Sec. 5] by restricting $L^*$ to the submanifold defined by (the Euler equations)

$$(2.3.1) \qquad g^{ij} x_j^k - p_k^i = 0$$

obtaining

$$L\hat{} = -H + \sum_k p_k^i (x^k)_i$$

where

$$H = \frac{1}{2} \sum_k [p_{ik} p_k^i + m^2 x^k x^k].$$

This has the same extremals as $L$.

When $n = 2$ we simplify the notation by letting $x^1 = x, x^2 = y, p_1^i = p^i, p_2^i = q^i$.

So

$$
\begin{aligned}
L\hat{} &= -\frac{1}{2}[p_i p^i + q_i q^i + m^2(x^2+y^2)] + p^i(x)_i + q^i(y)_i \\
&= -H + p^i(x)_i + q^i(y)_i
\end{aligned}
\tag{2.4}
$$

where

$$
H = \frac{1}{2} p_i p^i + \frac{1}{2} q_i q^i + \frac{1}{2} m^2(x^2+y^2). \tag{2.5}
$$

NOTE: The Theorem 2.6 and those lemmas used in its proof require only that

$$
H = \frac{1}{2} p_i p^i + \frac{1}{2} q_i q^i + h(x,y). \tag{2.5.1}
$$

The extremals of (2.4) are submanifolds of some large space, but they lie in the submanifold defined by (2.3.1) where $x, y, t^1, \ldots, t^4, p_1, \ldots, p_4, q_1, \ldots, q_4$ are coordinates. These have the same transformation properties as the original $x, y, t^1, \ldots (x)_1, \ldots, (x)_4, (y)_1, \ldots, (y)_4$. Thus the extremals can be identified with submanifolds of $J^1(\mathbb{R}^4, Q)$ where $Q = \mathbb{R}^2$, and $p_i$ taken as an abbreviation for $(x)_i$. The currents (2.1) are 4-forms in $J^1(\mathbb{R}^4, Q)$. The components $J^i$ are functions of the 14 variables.

A final notation. Sometimes it will help to denote $x$ by $t^5$ and $y$ by $t^6$. A sum with Greek indices shall be understood to go from 1 to 6: $a_\lambda t^\lambda = a_1 t^1 + \ldots + a_4 t^4 + a_5 x + a_6 y$. We also use $p_\lambda$ and $q_\lambda$ and $p_5 = -1, p_6 = 0, q_5 = 0, q_6 = -1$.

We treat first the real 2-dimensional Klein-Gordon field because the computations can be more easily adapted to the 1-dimensional case (sec. 9 below) than the other way around.

2.6. THEOREM. *The most general dynamic current*

$$
\delta = J^1 \mathrm{d}^{234} - J^2 \mathrm{d}^{134} + \ldots - \ldots
$$

*is obtainable as follows. Select functions $A^1, \ldots, A^4, B^1, \ldots, B^4$ which depend only on $t^1, t^2, t^3, t^4$; and functions $u, v$ which depend only on the t's, x, and y such that*

$$
\frac{\partial A_k}{\partial t^\ell} + \frac{\partial A_\ell}{\partial t^k} = 2 g_{k\ell}\left(\frac{1}{2}\frac{\partial A^i}{\partial t^i} + \frac{\partial u}{\partial x}\right) \tag{2.6.1}
$$

$$
\frac{\partial B_k}{\partial t^\ell} + \frac{\partial B_\ell}{\partial t^k} = 2 g_{k\ell}\left(\frac{1}{2}\frac{\partial B^i}{\partial t^i} + \frac{\partial v}{\partial y}\right) \tag{2.6.2}
$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0. \tag{2.6.3}$$

*Select four functions* $G^1$, $G^2$, $G^3$, $G^4$ *depending only on* $t^1, \ldots, t^4$, $x, y$ *such that*

$$\frac{\partial G^k}{\partial x} = -u^k - A^k H_x \tag{2.6.4}$$

$$\frac{\partial G^k}{\partial y} = -v^k - B^k H_y \tag{2.6.5}$$

$$\frac{\partial G^k}{\partial t^k} = uH_x + vH_y \tag{2.6.6}$$

*where* $H_x = \partial H/\partial x$, *etc.*

*Select a 2-form* $\xi$ *in the variables* $x, y, t$, *and consider*

$$\Gamma = G^1\, \mathrm{d}^{234} - G^2\, \mathrm{d}^{134} + G^3\, \mathrm{d}^{124} - G^4\, \mathrm{d}^{123} + \mathrm{d}\xi. \tag{2.6.7}$$

*Define* $C^{\lambda\mu\nu}(= -C^{\mu\lambda\nu} = C^{\mu\nu\lambda})$ *by*

$$\Gamma = \Sigma\, \epsilon_{\lambda\mu\ldots\tau}\, C^{\lambda\mu\nu}\, \mathrm{d}^{\rho\sigma\nu} \tag{2.6.8}$$

*where this sum is extended over those* $(\lambda, \mu, \nu, \rho, \sigma, \tau)$ *for which* $\lambda < \mu < \nu$, $\rho < \sigma < \tau$. *Here* $\epsilon_{\lambda\mu\ldots\tau} = +1, -1, 0$ *according to whether* $(\lambda, \mu, \ldots, \tau)$ *is an even, an odd, or no permutation of* $(1, 2, \ldots, 6)$.

Then let

$$J^i = \frac{1}{2} A^i p_j p^j - p^i A_j p^j + \frac{1}{2} B^i q_j q^j - q^i B_j q^j + up^i + vq^i + C^{i\lambda\mu} p_\lambda q_\mu. \tag{2.6.9}$$

For an example, see Section 7 below.

We now present three lemmas which help in the proof of (2.6).

2.7. LEMMA. *The 3-form* (2.1) *is dynamic if and only there exist* $\mu^k (k = 1, 2)$ *such that*

$$\frac{\partial J_m}{\partial p^k} + \frac{\partial J_k}{\partial p^m} = 2\mu^1 g_{mk}, \tag{2.7.1}$$

(2.7.2) $$\frac{\partial J_m}{\partial q^k} + \frac{\partial J_k}{\partial q^m} = 2\mu^2 g_{mk},$$

*and*

(2.7.3) $$\frac{\partial J^i}{\partial t^i} + \frac{\partial H}{\partial p^i}\frac{\partial J^i}{\partial x} + \frac{\partial H}{\partial q^i}\frac{\partial J^i}{\partial y} = \mu^1\frac{\partial H}{\partial x} + \mu^2\frac{\partial H}{\partial y}.$$

2.8. LEMMA. *The $J^i$ satisfy* (2.7.1) *and* (2.7.2) *if and only if*

(2.8.1) $$J^i = \frac{1}{2}A^i p_j p^j - p^i A_j p^j + \frac{1}{2}B^i q_j q^j - q^i B_j q^j$$
$$+ C^{ijk} p_j q_k + m^{i\ell} p_\ell + n^{i\ell} q_\ell + u p^i + v q^i + w^i$$

*where*

(2.8.2) *these variables $A, \ldots, w$ depend only on $x, y$, and $t$ (meaning $t^1, \ldots, t^4$)*

(2.8.3) $-m^{i\ell} = m^{\ell i},\ n^{i\ell} = -n^{\ell i}$

(2.8.4) $C^{ijk} = -C^{jik} = C^{jki}$.

2.9. LEMMA. *Suppose that $J^i$ satisfies the conclusion of* (2.8). *Let*

$$C^{jk5} = n^{jk}$$
$$C^{jk6} = -m^{jk}$$
$$C^{j56} = w^j$$

*Then* (2.7.3) *will hold if and only if*

(2.9.1) $$A_{k\ell} + A_{\ell k} = 2g_{k\ell}\left(\frac{1}{2}A^i_i + u_x\right)$$

(2.9.2) $$B_{k\ell} + B_{\ell k} = 2g_{k\ell}\left(\frac{1}{2}B^i_i + v_y\right)$$

(2.9.3) $$\frac{\partial A^i}{\partial x} = \frac{\partial A^i}{\partial y} = \frac{\partial B^i}{\partial x} = \frac{\partial B^i}{\partial y} = 0$$

(2.9.4) $$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial C^{\lambda k5}}{\partial t^\lambda} = -v^k - B^k H_y \tag{2.9.5}$$

$$\frac{\partial C^{\lambda k6}}{\partial t^\lambda} = u^k + A^k H_x \tag{2.9.6}$$

$$\frac{\partial C^{\lambda 56}}{\partial y^\lambda} = uH_x + vH_y \tag{2.9.7}$$

$$\frac{\partial C^{\lambda ij}}{\partial t^\lambda} = 0. \tag{2.9.8}$$

Here $A_{k\ell}$ means $\partial A_k/\partial t^\ell$, where of course $A_k = g_{km}A^m$, etc. Similarly, $v^k$ means $g^{km}\,\partial v/\partial t^m$. We recall that $\lambda$ is summed from 1 to 6, and $t^5 = x$, $t^6 = y$. $H_x$ is $\partial H/\partial x$.

Consider the case

$$h(x, y) = \frac{1}{2}\, m^2(x^2 + y^2). \tag{2.9.9}$$

When $m \neq 0$, the the equations (2.6.1) - (2.6.6) force the right hand sides of (2.6.1, 2.6.2) to be 0, thereby limiting $A$ and $B$ to be Poincaré vector fields. On the other hand, when $m = 0$, and two arbitrary conformal vector fields $A$ and $B$ (see 6.1) are given, then $u, v, G^1, \ldots, G^4$ can be found satisfying (2.6.1) - - (2.6.6). One may take $u = (-\lambda + a_j t^j)\,x, v = (-\mu + b_j t^j)\,y, G^k = -1/2\, a^k x^2 - -1/2\, b^k y^2$ and $A$ as in (9.3.7) ($\mu$ and $b = 0$), $B$ analogously for a properly conformal solution.

## 3. DEDUCING (2.6) FROM THESE LEMMAS

We first assume $\delta$ is dynamic and work our way to the relations (2.6.1) (2.6.8). From (2.7) and (2.8) we have (2.8.1) - (2.8.4), (2.9.1) - (2.9.8). From (2.8.1) and the definitions of the $C's$, we have (2.6.9). Properties (2.6.1) and (2.6.2) are (2.9.1) and (2.9.2) while (2.9.3) implies that the $A's$ and $B's$ depend only on the $t's$.

Since we are proving the converse of the statement of (2.6), we define $\Gamma$ using (2.6.8). We must deduce (2.6.6, 2.6.3 - 2.6.5) as well as that those $G's$ depend only on $t^1$ to $t^6$.

We abbreviate $\partial C^{\lambda\mu\nu}/\partial t^\xi$ by $C^{\lambda\mu\nu}_\xi$, and we abbreviate $\partial C^{\lambda\mu\nu}_\xi/\partial t^\eta$ by $C^{\lambda\mu\nu}_{\xi\eta}$. From (2.6.8) we obtain

$$(3.1) \qquad \begin{aligned} d\Gamma = {} & C_\lambda^{\lambda 56}\,d^{1234} - C_\lambda^{\lambda 15}\,d^{2346} + C_\lambda^{\lambda 16}\,d^{2345} \\ & + C_\lambda^{\lambda 25}\,d^{1346} - C_\lambda^{\lambda 26}\,d^{1345} \\ & - C_\lambda^{\lambda 35}\,d^{1246} + C_\lambda^{\lambda 36}\,d^{1245} \\ & + C_\lambda^{\lambda 45}\,d^{1236} - C_\lambda^{\lambda 46}\,d^{1235} \\ & \pm \text{ terms of the form } C_\lambda^{\lambda ij}\,d^{k\ell 56}. \end{aligned}$$

These last terms are 0 by (2.9.8). Hence

$$\begin{aligned} dd\Gamma = {} & C_{\lambda 5}^{\lambda 56}\,d^{12345} + C_{\lambda 6}^{\lambda 56}\,d^{12346} - C_{\lambda 1}^{\lambda 15}\,d^{12346} + C_{\lambda 5}^{\lambda 15}\,d^{23456} \\ & + C_{\lambda 1}^{\lambda 16}\,d^{12345} + C_{\lambda 6}^{\lambda 16}\,d^{23456} - C_{\lambda 2}^{\lambda 25}\,d^{12346} \\ & + C_{\lambda 2}^{\lambda 26}\,d^{12345} - C_{\lambda 3}^{\lambda 35}\,d^{12346} \\ & + C_{\lambda 3}^{\lambda 36}\,d^{12345} - C_{\lambda 4}^{\lambda 45}\,d^{12346} + C_{\lambda 4}^{\lambda 46}\,d^{12345} + \ldots . \end{aligned}$$

Here we have written down only those terms with $d^{12345}$, $d^{12346}$, and $d^{23456}$. The coefficient of the last is

$$C_{\lambda 5}^{\lambda 15} + C_{\lambda 6}^{\lambda 16}.$$

Since this is 0 we have

$$\frac{\partial C_\lambda^{\lambda 16}}{\partial y} = \frac{\partial C_\lambda^{\lambda 51}}{\partial x} .$$

Thus there is an $F^1$ depending only on $t, x, y$ such that

$$C_\lambda^{\lambda 16} = \frac{\partial F^1}{\partial x} , \qquad C_\lambda^{\lambda 51} = \frac{\partial F^1}{\partial y} .$$

We obtain $F^1, F^2, F^3, F^4$ such that

$$(3.2) \qquad C_\lambda^{\lambda k6} = \frac{\partial F^k}{\partial x} , \qquad C_\lambda^{\lambda 5k} = \frac{\partial F^k}{\partial y} .$$

The coefficient of $d^{12345}$ is

$$C_{\lambda 5}^{\lambda 56} + C_{\lambda k}^{\lambda k6} \quad \text{or} \quad C_{\lambda 5}^{\lambda 56} + \frac{\partial}{\partial t^k}\left(\frac{\partial F^k}{\partial x}\right).$$

Now this is zero, so

$$\frac{\partial}{\partial x}\left(C_\lambda^{\lambda 56} + \frac{\partial F^k}{\partial t^k}\right) = 0.$$

This is also true with $x$ replaced by $y$, so

$$C_\lambda^{\lambda 56} + \frac{\partial F^k}{\partial t^k} = \textit{function of } t \textit{ only} = \varphi(t). \tag{3.3}$$

Now define

$$F = F^1\,\mathrm{d}^{234} - F^2\,\mathrm{d}^{134} + F^3\,\mathrm{d}^{124} - F^4\,\mathrm{d}^{123},$$

and compute $\mathrm{d}F$. Using (3.1, 3.2) and (3.3) one obtains easily that

$$\mathrm{d}F = -\,\mathrm{d}\Gamma + \varphi(t)\,\mathrm{d}^{1234}.$$

Hence $\Gamma = -F + \mathrm{d}\xi$. Putting $F = -G$ we have (2.6.6), and also that the $G$'s depned only on $x$, $y$, and $t$.

We come to (2.6.3).

$$\frac{\partial G^k}{\partial x} = -\frac{\partial F^k}{\partial x} = -C_\lambda^{\lambda k 6} = -u^k - A^k H_x$$

by (3.2) and (2.9.6). This establishes (2.6.3). The remaining (2.6.4, 2.6.5) are derived in the same way. Hence we have proved (2.6) in the one direction.

For proving the other direction, we assume that these $A, B, \ldots$ mentioned in (2.6) are given with the properties enumerated, and we must prove that $\delta$ is dynamic.

We are given that (2.6.7) holds. We compute $\mathrm{d}\Gamma$ and use (2.6.4 - 2.6.6). This tells us (2.9.5 - 2.9.8). The assumptions in (2.6) about $A, B, u, v$ give us (2.9.1 - 2.9.4). Then by (2.9) we obtain (2.7.3).

We are also given (2.6.9). We define $m, n, w$ as in (2.9); and this implies (2.8.1 - 2.8.4). Thus we have (2.7.1) and (2.7.2) also. We deduce from (2.7) that $\delta$ is dynamic.

In the course of constructing a dynamic form $\delta$ according to the recipe of (2.6), a 3-form $\xi$ in the variables $t (= t^1, t^2, t^3, t^4)$, $x, y$ is selected. This $\xi$ does not have to be related to the other quantities $(A^1, \ldots, G^4)$.

3.4. PROPOSITION. *Let $\delta$ be a dynamic form involving a certain $\xi$. Let $\delta'$ be another constructed with the same $A^1, \ldots, G^4$ but some other $\xi$, say $\xi'$. Then*

$$\delta' - \delta \equiv \mathrm{d}\xi' - \mathrm{d}\xi \ \text{ modulo } X, Y$$

*where (see 7.3 below).*

(3.4.1) $$X = \mathrm{d}x - p_i\,\mathrm{d}t^i$$

(3.4.2) $$Y = \mathrm{d}y - q_i\,\mathrm{d}t^i.$$

Thus $\delta' - \delta$ is dynamically null (see §9). In otherwords, the arbitrariness in $\xi$ affects $\delta$ only by a summand which is dynamically null. To prove this, we need only take $A^1 = \ldots = G^4 = 0$ and show that the resulting $\delta$ has

$$\delta \equiv \mathrm{d}\xi \ \textit{modulo} \ X, Y.$$

For example, let $\mathrm{d}\xi = \mathrm{d}^{125}$. Then $\mathrm{d}\xi = -C^{346}\,\mathrm{d}^{125}$ and $C^{346} = -1$ and $C^{\lambda\mu\nu} = 0$ for $\{\lambda, \mu, \nu\} \neq \{3, 4, 6\}$. So only $J^3$ and $J^4$ are non-zero.

$$J^3 = C^{3\lambda\mu} p_\lambda q_\mu = -p_4 q_6 + p_6 q_4 = p_4$$

$$J^4 = C^{4\lambda\mu} p_\lambda q_\mu = C^{436} p_3 q_6 + C^{463} p_6 q_3 = -p_3.$$

Then $$\begin{aligned}\delta &= p_4\,\mathrm{d}^{124} + p_3\,\mathrm{d}^{123} = (p_3\,\mathrm{d}^3 + p_4\,\mathrm{d}^4)\,\mathrm{d}^{12}\\ &= (p_1\,\mathrm{d}^1 + p_2\,\mathrm{d}^2 + p_3\,\mathrm{d}^3 + p_4\,\mathrm{d}^4)\,\mathrm{d}^{12} = (\mathrm{d}x - X)\,\mathrm{d}^{12}\\ &= \mathrm{d}^{12}\,\mathrm{d}x - X\,\mathrm{d}^{12} = \mathrm{d}\xi - X\,\mathrm{d}^{12} \equiv \mathrm{d}\xi \ \textit{modulo} \ X.\end{aligned}$$

There are two other cases $\mathrm{d}\xi = \mathrm{d}^{234}$ and $\mathrm{d}^{456}$. In the last case, $\delta = (\mathrm{d}x - X)(\mathrm{d}y - Y)\,\mathrm{d}^4 \equiv \mathrm{d}^4\,\mathrm{d}x\,\mathrm{d}y$ modulo $X$, $Y$ as the reader may verify.

## 4. PROOF OF 2.7

Let $\epsilon$ be any 4-form in our $\mathbb{R}^{14}$ and let $U_1, U_2, U_3, U_4$ be four vector fields. The evaluation on of $\epsilon$ on $U_1 \otimes \ldots \otimes U_4$ will be denoted by

$$\langle \epsilon; U_1, U_2, U_3, U_4 \rangle.$$

It can be evaluated by forming $U_4 \lrcorner\, \epsilon$, $U_3 \lrcorner\, U_4 \lrcorner\, \epsilon, \ldots, U_1 \lrcorner\, U_2 \lrcorner\, U_3 \lrcorner\, U_4 \lrcorner\, \epsilon = \langle \epsilon; U_1, U_2, U_3, U_4 \rangle$.

If $U$ is a vector field, $U = \dfrac{\partial}{\partial t^1} + V\dfrac{\partial}{\partial x}$ for example, then $U[f]$ stands for $\dfrac{\partial f}{\partial t^1} + V\dfrac{\partial f}{\partial x}$, and so forth.

For each $i$ let

(4.0.1) $$U_i = \frac{\partial}{\partial t^i} + \frac{\partial H}{\partial p^j_K}\,\frac{\partial}{\partial x^K} + U^j_{iK}\,\frac{\partial}{\partial p^j_K}\,.$$

Here $H$ might be as in (2.5), but the $U^j_{iK}$ are just parameters.

4.1. LEMMA. *Let $\delta$ be a 3-form in $\mathbb{R}^{14}$. Then $\delta$ is dynamic if and only if*

$\langle \mathrm{d}\delta, U_1, U_2, U_3, U_4\rangle$ *is* 0 *whenever the parameters* $U^j_{iK}$ *make vanish all the expressions*

$$(4.1.1)\qquad U_k\left[\frac{\partial H}{\partial p^\ell_M}\right]-U_\ell\left[\frac{\partial H}{\partial p^k_M}\right]$$

*and*

$$(4.1.2)\qquad U^i_{iM}+\frac{\partial H}{\partial x^M}\qquad\qquad (\text{sum on } i;\ M=1,2),$$

*provided that the rank of the Hessian matrix*

$$\frac{\partial^2 H}{\partial p^i_K\,\partial p^j_L}$$

*is at least* 6.

*Proof.* A reference to [2, 6.2] and simple computations of $X^\lambda$, $\mathrm{d}X^\lambda$, (see [2]) shows that the vanishing of (4.1.1) and (4.1.2) is a necessary and sufficient condition that there should be an extremal tangent to $U_1, \ldots, U_4$. So let $\delta$ be dynamic. Then if (4.1.1, 4.1.2) are 0 then $\langle \mathrm{d}\delta; U_1, \ldots, U_4\rangle$ must be 0. If $\delta$ is not dynamic then $\langle \mathrm{d}\delta; U_1, \ldots, U_4\rangle$ will be not 0 for some four vectors $U_1, \ldots, U_4$ which are tangent to an extremal and for which (4.1.1, 4.1.2) will therefore vanish.

The 6 in this statement generalizes to $3n$ for $n \neq 2$.

For the case of a current $\delta$ (3.1), $\langle \delta; U_1, U_2, U_3, U_4\rangle$ takes on the form

$$(4.1.3)\qquad U_i[J^i].$$

For $H$ as in (2.5), the expressions (4.1.1) take the form

$$(4.1.4)\qquad U_{k\ell M}-U_{\ell kM}$$

where

$$U_{k\ell M}=U^j_{kM}g_{j\ell}.$$

Hence $U_i[J^i]$ is a linear function (call it $f$) of the variables $U^j_{kM}$ (or if preferable, the $U_{k\ell M}$) which vanishes when certain other first degree polynomials in those variables vanish, namely the (4.1.1) and (4.1.2). Denote the latter briefl by $g_1, \ldots, g_N$. A simple convexity argument shows that there are constant (that is, expressions independent of the $U^j_{kM}$) $c_1, \ldots, c_N$ such that $f = c_1 g_1 + \ldots +$

$+ c_N g_N$. We formulate the result.

4.2. LEMMA. *Let $\delta$ be as in (3.1) and $H$, as in (2.5). Then $\delta$ is dynamic if and only if there are quantities $L^{k\ell}_M$, $\mu^M$ independent of the $U^j_{kM}$ for which*

(4.2.1) $$U_i[J^i] = L^{k\ell}_M(U_{k\ell M} - U_{\ell kM}) + \mu^M(U^i_{iM} + m^2x^M)$$

*and*

(4.2.2) $$L^{k\ell}_M = -L^{\ell k}_M.$$

We do intend a sum on $M, i, k, \ell$ in (4.2.1). Item (4.2.2) is a trivial but convenient embellishment.

If one sets equal to 0 all the $U^j_{iK}$, then (4.2.1) reduces to (2.7.3). If one equates coefficients of $U^1_{1K}$ in (4.2.1), then one obtains

$$\frac{\partial J_1}{\partial p^1_K} = \mu^K g_{11}.$$

The coefficients of $U^1_{2K}$ give

$$\frac{\partial J_2}{\partial p^1_K} + \frac{\partial J_1}{\partial p^2_K} = 0.$$

Here we assume the metric diagonal. Thus (2.7.1) is established. By interchanging $p$ and $q$, we also have (2.7.2) and the proof is complete.

## 5. A PROOF OF 2.8

Let $m \geqslant 3$.

The significance of (2.7.1) all by itself is given by the following.

5.1. THEOREM. *Let $g_{ij}$ be a symmetric, regular $m \times m$ constant matrix. Let $J_1, \dots, J_m$ be differentiable functions defined in $\mathbb{R}^m$ such that in terms of cartesian coordinates $z^1, \dots, z^m$*

(5.1.1) $$\frac{\partial J_i}{\partial z^j} + \frac{\partial J_j}{\partial z^i} = 2\mu g_{ij} \qquad (i, j = 1, \dots, m).$$

*Then there exist constants $a_i, m_{ij}, e_i$ and $u$ where $i, j = 1, \dots, m$; $m_{ij} = -m_{ji}$ and*

(5.1.2) $$J_i = 1/2\, a_i z^j z_j - a_j z^j z_i + m_{ij} z^j + uz_i + e_i.$$

Here $z_j$ is to be interpreted by the usual rules for raising and lowering indices.

Conversely, (5.1.2) implies (5.1.1) with

(5.1.3) $$\mu = u - a_i z^i.$$

The proof of this converse is easy and need not be presented.

The proof of (5.1) begins with a lemma.

5.2. LEMMA. *If* (5.1.1) *holds with* $m > 2$ *then the partial derivatives*

$$\mu_{ij} \equiv \frac{\partial^2 \mu}{\partial z^i \, \partial z^j}$$

*all vanish.*

*Proof.* We assume, as we may, that $g$ is diagonal. From $J_{13} + J_{31} = 0$ we get $J_{133} + J_{331} = 0$. Since $J_{33} = \mu g_{33}$ we have

(5.2.1) $$J_{133} + \mu_1 g_{33} = 0.$$

Of course also $J_{233} + \mu_2 g_{33} = 0$. From this we have $\mu_{21} g_{33} + J_{2331} + J_{1332} + + \mu_{12} g_{33} = 0$ or $2 g_{33} \mu_{12} = -(J_{21} + J_{12})_{33}$ which is 0. Thus $\mu_{12} = 0$.

Quoting (5.2.1) again gives $g_{33}\mu_{11} = -J_{1331} = (-J_{11})_{33} = (-\mu g_{11})_{33}$ whence

$$\frac{\mu_{11}}{g_{11}} = -\frac{\mu_{33}}{g_{33}}$$

from which follow

$$\frac{\mu_{33}}{g_{33}} = -\frac{\mu_{22}}{g_{22}}, \qquad \frac{\mu_{22}}{g_{22}} = -\frac{\mu_{11}}{g_{11}}.$$

So $\mu_{11}$ is 0 and (5.2) is proved.

So now we know $\mu$ has the form (5.1.3). Define

$$K_i = J_i - \frac{1}{2} a_i z^k z_k + a_k z^k z_i - u z_i.$$

It is easy to see that

$$\frac{\partial K_i}{\partial z^j} + \frac{\partial K_j}{\partial z^i} = 0.$$

These equations (due to Killing) characterize infinitesimal isometries (in the sense of $g_{ij}$). From this we get the rotational part $m_{ij}$ and the translational part $e_i$:

$$K_i = m_{ij} z^j + e_i$$

and (5.1) is proved.

The equations (5.1.1) characterize, and the formula (5.1.2) represents, an infinitesimal conformal transformation. Thus (5.1) is an infinitesimal version of Liouville's theorem because it is easy to show that a vector field like (5.1.2) is an infinitesimal similarity conjugated by an inversion (all understood in terms of the given metric).

Let

$$G^k_{\ell rs} = \delta^k_\ell g_{rs} - \delta^k_r g_{\ell s} - \delta^k_s g_{r\ell}. \tag{5.3}$$

Then the quadratic part of $J^k$ (note the raised index) is

$$\frac{1}{2} G^k_{\ell rs} a^\ell z^r z^s. \tag{5.3.1}$$

When $z$ is $p$ or $q$ we will abbreviate

$$\frac{1}{2} G^k_{\ell rs} p^r p^s = \frac{1}{2} \delta^k_\ell p_r p^r - p^k p_\ell \quad \text{by} \quad P^\ell_k \tag{5.3.2}$$

and

$$\frac{1}{2} G^k_{\ell rs} q^r q^s \qquad \text{by} \qquad Q^k_\ell .$$

We begin the proof of (2.8). From (2.7.1) and (5.1) we obtain

$$J_i = \frac{1}{2} a_i p_j p^j - p_i a_j p^j + m_{ij} p^j + \mu p_i + e_i \tag{5.4}$$

where $a$, $m$, $u$, $e$ depend only on $t, x, y$, and unfortunately, $q$.

Let $\Box$ be the operator $g^{k\ell} \partial^2/\partial p^k \partial p^\ell$. Applying it to both sides of (2.7.2) shows that $\Box J_m$ also satisfies (2.7.2) although with a different $\mu$. Now $\Box J_m = 2a_m$ (the 2, by the way, is 2 less than the dimension of space time in general). Then we can apply (5.1) and obtain

$$a^k = Q^k_\ell b^\ell + n^{kj} q_j + v q^k + f^k$$

where these $b, n, v, f$ depend only on $t, x, y$. Therefore

$$J^i = P^i_k Q^k_\ell b^\ell + a \text{ cubic in } p \text{ and } q.$$

By symmetry,

$$J^i = Q^i_k P^k_\ell b^\ell + a \text{ cubic in } p \text{ and } q.$$

Hence $P^i_k Q^k_\ell b^\ell = Q^i_k P^k_\ell c^\ell$, from which

$$P^i_k G^k_{\ell rs} b^\ell = G^i_{krs} P^k_\ell c^\ell.$$

For $r = 1$, $s = 2$, $i = 3$ one has $G^i_{\ell rs} = 0$, so $P^3_k G^k_{\ell 12} b^\ell = 0$. Thus $P^3_k h^k = 0$ where $h^1 = b_2$, $h^2 = b_1$, $h^3 = h^1 = 0$. So $\Box (P^3_k h^k) = 0$, and $h = 0$ whence $b_1 = 0$. In the same way each $b^\ell = 0$, and so (5.4) holds with

$$(5.4.1) \qquad a^k = n^{kj} q_j + vq^k + f^k.$$

5.5. LEMMA. *These n's and v's are* 0.
Note: These are not the $n$'s and $v$'s of (2.8.1).

We begin to prove this by a simple observation.

5.5.1. PROPOSITION. $e_1, \dots, e_4$ *(see 5.4) satisfy* (2.7.2) *with a suitable* $\mu$, *and so*

$$(5.5.2) \qquad e^i = Q^i_k \beta^k + \nu^{ik} q_k + wq^i + \varphi^i$$

*where* $\beta^k, \dots, \varphi^i$ *depend only on* $t, x, y$.

5.5.3. PROPOSITION. *For each* $\ell$, $m^{i\ell} + ug^{i\ell} (i = 1, \dots, 4)$ *satisfies* (2.7.2) *with a suitable* $\mu$ *and so*

$$(5.5.4) \qquad m^{i\ell} + ug^{i\ell} = Q^i_k b^{\ell k} + n^{\ell ik} q_k + v^\ell a^i + f^{\ell i}$$

*where these coefficients depend only on* $t, x, y$.

We now sketch the proof of (5.5) which is merely an exercise in algebra.

Insert (5.5.2) and (5.5.4) into (5.4) giving an expression for $J_i$ of the form

$$\Psi_i(p, q, n, v, f, \dots) = J_i,$$

where the dots represent the coefficients of (5.5.4). We impose on $J$ the conditions (2.7.2) and it tells us that for $i \neq h$, one has $P^k_i N_{kh} + P^k_h N_{ki} = 0$, where $N_{kh} = n_{kh} + vg_{kh}$. This gives $G^k_{i12} N_{kh} + G^k_{h12} N_{ki} = 0$ which for $i = 1$ and $h = 3$ shows $N_{23} = 0$. Assuming $g$ diagonal gives us $n_{23} = 0$ or $n = 0$. Then taking $i = 1$, $h = 2$ gives $v = 0$.

It appears that $\Psi_i(p, q, 0, 0, f, \dots)$ has no $ppq$ terms. By this we mean terms like

$$\text{const. } p_i p_j q_k.$$

Hence we may delete the $ppq$ terms. The result is that $J_i$ has the form (2.8.1).

We still have to show (2.7.1, 2.7.2). For the moment we can discard the $u$ and $v$ terms and just decompose the $m^{i\ell}$ into symmetric and skew parts

$$m^{i\ell} = \alpha^{i\ell} + \beta^{i\ell}.$$

A simple argument shows that $\beta^{i\ell}$ is a scalar matrix. A similar argument works for $n$ and $C$. Thus (2.7.1, 2.7.2) implies (2.8.1 - 2.8.4). The opposite implication is trivial. (2.8) is proved.

## 6. PROOF AND DISCUSSION OF 2.9

Our proof of (2.9) consists in inserting (2.8.1) into (2.7.3), and breaking the result up into homogeneous parts.

Looking at the *ppp* terms gives $\partial A/\partial x = 0$. Looking at the *ppq* terms tells one that $\partial A/\partial y = 0$. Thus we have (2.9.3).

The *pp* terms gives (2.9.1). The *pq* terms gives two results: One is (2.9.4) and the other is (2.9.8).

When equating first degree or zero degree terms, the form of $\mu^1$ given by (5.1.3) comes into play. The $p$ terms gives (2.9.6).

The $q$ terms do not give the dual the $p$ terms, because we are looking at (2.8.1) and not (2.8.2). The $q$ terms give (2.9.5).

The terms containing neither $p$ nor $q$ give (2.9.7). This ends our demonstration of (2.9).

One should appreciate the meaning of (2.9.1 - 2.9.3).

6.1 PROPOSITION. *$A$ and $B$ are infinitesimal conformal vector fields in space-time. They are independent of the field $(x, y)$.*

(2.9.1) is an instance of (5.1.1), so the $A_i$ must be of the form (5.1.2) where the coefficients are constants (and $z$ is replaced by $t$. The coefficients appearing in the (5.1.2) representation of $A_i$ should not be confused with the $a$, $m$, $u$, $e$ appearing elsewhere, such as in (5.4). The latter need not be constant and generally depend on $t, x, y$).

The emergence of these conformal vector fields $A$ and $B$ suggests wondering if such fields are (infinitesimal) symmetries of the Klein-Gordon fields. We consider this in Section 9.

## 7. SOME SPECIFIC DYNAMIC FORMS

7.1. PROPOSITION. *Let $H$ be as in (2.5). Select eight constants $A^1, \ldots, A^4$, $B^1, \ldots, B^4$. Let*

$$(7.2) \qquad J^k = \frac{1}{2} A^k p_i p^i - p^k A^i p_i - \frac{1}{2} m^2 x^2 A^k + \frac{1}{2} B^k q_i q^i - q^k B^i q_i - \frac{1}{2} m^2 y^2 B^k.$$

Then $J$ defines a dynamic current.

To see this, let $G^k = -\frac{1}{2} m^2 x^2 A^k - \frac{1}{2} m^2 y^2 B^k$. This satisfies (2.6.1 - - 2.6.6). It makes $C^{k56} = G^k$ and $C^{\lambda\mu\nu} = 0$ for $\lambda, \mu, \nu$ not permutations of $k$, 5, 6, This produces a $J$ as in (7.2).

These dynamic forms are the specimens which will provide the counterexample to equality (1.3).

We repeat the definition of Hamiltonic forms, changing $U$ to $-U$ as is sometimes done. $\varphi$ is Hamiltonic if $\varphi = \psi + U \lrcorner \alpha$ where $\psi = -\mathcal{L}_U \alpha$. Then $d\varphi = = d\psi + d(U \lrcorner \alpha) = -[U \lrcorner d\alpha + d(U \lrcorner \alpha)] + d(U \lrcorner \alpha)$ so

$$(7.3) \qquad d\varphi = -U \lrcorner d\alpha.$$

If $\Lambda$ is a Lagrangean density of the reduced type (as (2.4) is), say

$$\Lambda = -H + p^i (x)_i,$$

then $\Lambda$ has the same extremals as the 4-form

$$(7.4) \qquad \alpha = -H d^{1234} + dx(p^1 d^{234} - p^2 d^{134} + \ldots).$$

This is easily seen by comparing the Euler-Lagrange-Hamilton equations [2, p. 7; pp. 21 - 22].

An almost infallible mnemonic device is that

$$\alpha = L\, d^{1234} + X^k F_K$$

where

$$(7.5) \qquad X^k = dx^K - (x^K)_i\, dt^i$$

and $F^1, \ldots, F^4$ are suitable 3-forms.

The application of this idea to the $L\hat{}$ (2.4) gives the $\alpha$ in (8.1) below.

Let us call a dynamic form $\delta$ remotely Hamiltonic if it is a sum

$$(7.6) \qquad \delta = \varphi + \zeta + d\eta$$

where $\varphi$ is Hamiltonic, $\zeta$ vanishes on all extremals, and $\eta$ is any 2-form. Then the question about (1.3) is answered as follows.

7.7. THEOREM. *The dynamic form* (7.1) *is not remotely Hamiltonic if* $A \neq B$.

## 8. PROOF OF 7.7

To prove this, we must find the most general 3-form $\zeta$ which vanishes on all the extremals for

(8.1) $$\alpha = -H\mathrm{d}^{1234} + \mathrm{d}x(p^1\mathrm{d}^{234} - p^2\mathrm{d}^{134} + \ldots) + \mathrm{d}y(q^1\mathrm{d}^{234} - q^2\mathrm{d}^{134} + \ldots),$$

which corresponds to (2.4) (see (7.4)).

We assume $H$ has the form (2.5) (actually, the next lemma remains true if the $m^2(x^2 + y^2)$ is replaced by any analytic function of $x$ and $y$).

8.2. LEMMA. *Let* $\zeta$ *be a 3-form which vanishes on each extremal for* (8.1). *Then*

$$\zeta = (\mathrm{d}x - p_i\,\mathrm{d}t^i)\Phi + (\mathrm{d}y - q_i\,\mathrm{d}t^i)\Psi + $$
$$+ \mathrm{d}p_i\,\mathrm{d}t^i(a^j\,\mathrm{d}p_j + b^j\,\mathrm{d}q_j) + \mathrm{d}q_i\,\mathrm{d}t^i(c^j\,\mathrm{d}p_j + e^j\,\mathrm{d}q_j)$$

*where* $\Phi$, $\Psi$ *are 2-forms and* $a^j, \ldots, e^j$ *are arbitrary functions.* $\Phi$ *does not contain* $\mathrm{d}x$, *and* $\Psi$ *does contain* $\mathrm{d}y$.

Such a form certainly does vanish on every extremal because $\mathrm{d}x - p_i\,\mathrm{d}t^i$, ... do [2, p. 15].

To prove (8.2) we begin with an arbitrary 3-form $\zeta$. We can certainly replace each $\mathrm{d}x$ by $p_i\,\mathrm{d}t^i$ and $\mathrm{d}y$ by $q_i\,\mathrm{d}t^i$ obtaining $\zeta'$ such that $\zeta' - \zeta$ vanishes on all extremals. Then we will show that $\zeta'$ has the above form with $\Phi = \Psi = 0$. We therefore assume

$$\zeta = Z^1\mathrm{d}^{234} - Z^2\mathrm{d}^{134} + Z^3\mathrm{d}^{124} - Z^4\,\mathrm{d}^{123}$$
$$+ \mathrm{d}p_L^i\, Z_{ijk}^L\,\mathrm{d}^{jk} + \mathrm{d}t^i\; Y_{ijk}^{JK}\,\mathrm{d}p_J^j\,\mathrm{d}p_K^k$$
$$+ W_{ijk}^{LMN}\,\mathrm{d}p_L^i\,\mathrm{d}p_M^j\,\mathrm{d}p_N^k,$$

*where* $Z_{ikj}^L = -Z_{ijk}^L$, $Y_{ikj}^{KJ} = -Y_{ikj}^{JK} = -Y_{ijk}^{JK}$, *etc.*

Here $p_1^i$ is what we abbreviated by $p^i$ and $p_2^i$ is $q^i$.

If $\zeta$ vanishes on all motion then

(8.2.1) $$\langle \zeta; U_i, U_j, U_k\rangle = 0$$

whenever (4.1.1) holds, and $U_i$, $U_j$, $U_k$ are any three of the four vector fields (4.0.1). (When helpful, we may forget about (4.1.2) because the fourth vector field can always be adjusted to insure (4.1.2) without affecting (8.2.1)). A routine calculation (including some raising and lowering of indices) shows that

$$\langle \zeta; U_1, U_2, U_3 \rangle = f = -Z_4 + Y$$

where

$$Y = 6\, U^K_{1k}\, U^L_{2\ell}\, U^M_{3m}\, W^{k\ell m}_{KLM} + \tag{8.2.2}$$
$$+\, 2 U^K_{1m}\, U^L_{2n}\, Y^{mn}_{3KL} + 2 U^K_{1m}\, Z^m_{K23}$$
$$+\, 2 U^K_{2m}\, U^L_{3n}\, Y^{mn}_{1KL} + 2 U^K_{2m}\, Z^m_{K31}$$
$$+\, 2 U^K_{3m}\, U^L_{1n}\, Y^{mn}_{2KL} + 2 U^K_{3m}\, Z^m_{K12}.$$

The reader will note that we wrote $U^K_{ij}$ instead of $U_{Kij}$ as in (4.14). This case of index raising is purely typographical, but moving the $i$ or $j$ involves the $g_{ik}$.

So $-Z_4 + Y = 0$ whenever

$$U^K_{ij} = U^K_{ji}. \tag{8.2.3}$$

Hence $Z_4$ must be 0 and so

$$Z_1 = Z_2 = Z_3 = Z_4 = 0. \tag{8.2.4}$$

$Y$ does not contain $U^K_{4j}$. So the $U^N_{j4}$ in $Y$ are not constrained by (8.2.3). Therefore

$$Y = 0 \quad \text{if} \quad (8.2.3) \text{ holds for} \quad i, j = 1, 2, 3. \tag{8.2.5}$$

The same thing is true for the linear terms in $Y$. In fact, for each $K$ the linear terms must add up to 0. The coefficient of $U^K_{11}$ is $Z^1_{K23}$, so $Z^1_{K23} = 0$. We also see

$$U^K_{12}\, Z^2_{K23} + U^K_{21}\, Z^1_{K31} = 0$$

whence $Z^2_{K23} = Z^1_{K13}$.

Here is a simple fact. If $V^i_{jk}$ are defined for $i, j, k$ (ranging from 1 to $P$, say) and $V^i_{jk} = V^i_{ik} - V^j_{jk} = 0$ when $i, j, k$ are all distinct (no sum on $i$ or $j$ here) and if $V^i_{jk} = -V^i_{kj}$ then

$$V^i_{jk} = \delta^i_j\, v_k - \delta^i_k\, v_j.$$

Therefore

$$Z^i_{Kjk} = \delta^i_j\, v_{Kk} - \delta^i_k\, v_{Kj}. \tag{8.2.6}$$

This holds for $i, j, k$ ranging from 1 to 4 because whichever integer is not among the $i, j, k$ could be given the role played by 4.

Apply the operator $\partial/\partial U^J_{14}$ to the quadratic part of $Y$. Dismissing a factor of 2 we obtain

(8.2.7) $$U^L_{2n}\, Y^{4n}_{3JL} + U^K_{3m}\, Y^{m4}_{2KJ} = 0.$$

The coefficient of $U^L_{22}$ must be 0, so $Y^{42}_{3JL} = 0$. Of course this implies that, $Y^{ij}_{kJL} =$ $= 0$ whenever $i, j, k$ are distinct.

Letting $U_{32} = U_{23}$ in (8.2.7) we note that the coefficient of $U^M_{23}$ is

$$Y^{43}_{3JM} + Y^{24}_{2MJ} = 0.$$

Thus $Y^{34}_{3MJ} = Y^{24}_{2MJ}$ or $Y^{jk}_{iMJ} = Y^{jk}_{jMJ}$ for any $i, j, k$ (no sum). As with (8.2.6) we can say

(8.2.8) $$Y^{ij}_{kJM} = \delta^i_k\, y^j_{JM} - \delta^j_k\, y^i_{JM}.$$

By taking $\partial/\partial U^J_{14}$ and then $\partial/\partial U^L_{21}$ of the cubic terms one obtains $U^M_{3m}\, W^{41m}_{JLM} =$ $= 0$ so that all the $W$ are 0. Assembling $\zeta$, using (8.2.6) and (8.2.8) one can easily show that it has the form claimed in (8.2), except for the claim that $\Phi$ need not contain $dx$, but this is easy to see and we thus end our proof of (8.2).

We will prove (7.7) only in the case of three-dimensional space-time. We will also let the $m$ in $H$ be zero. (If (7.7) is proved for such a current, then it must also be true for $m \neq 0$).

Denote the differential form derived from (7.2) by $\psi$:

$$\psi = J^1\, d^{23} - J^2\, d^{13} + J^3\, d^{12}.$$

Let

$$\begin{aligned}\zeta = \alpha\, dp_i\, dt^i + \beta\, dq_i\, dt^i + &\\ &+ (dx - p_1\, dt^i)(F^j\, dp_j + G^j\, dq_j + H_j\, dt^j + I\, dy)\\ &+ (dy - q_i\, dt^i)(J^j\, dp_j + K^j\, dq_j + L_j\, dt^j + M\, dx).\end{aligned}$$

This $\zeta$ is by (8.2) the most general 2-form which vanishes on all motions. We must show that we cannot have (7.6) unless $A = B$ in (7.2). Suppose therefore that (7.2) holds, or rather that $\psi - \varphi + \zeta = d\eta$. Then

(8.3) $$d\psi + U \lrcorner\, d\alpha + d\xi = 0$$

where $U$ is as in (7.3). Specifically, take

(8.4) $$U = C_k\, \frac{\partial}{\partial t^k} + D^k\, \frac{\partial}{\partial p^k} + E^k\, \frac{\partial}{\partial q^k} + V\, \frac{\partial}{\partial x} + W\, \frac{\partial}{\partial y}.$$

Then

(8.5) $$d\psi = [(A^1 p^j - A^j p^1) dp_j - A^j p_j\, dp^1 + (B^1 q^j - B^j q^1) dq_j - B^j q_j\, dq^1] d^{23} +$$

Here the $++$ are two more terms obtained by permuting the indices $1, 2, 3$ cyclically. This notation will be used often.

From

$$\alpha = -\frac{1}{2}(p_i p^i + q_i q^i)\,\mathrm{d}^{123} + \mathrm{d}x(p^1\,\mathrm{d}^{23} + +)$$
$$+\,\mathrm{d}y(q^1\,\mathrm{d}^{23} + +)$$

and (8.4) we obtain

$$\begin{aligned} U \lrcorner\, \mathrm{d}\alpha = &-(D^i p_i + E^i q_i)\,\mathrm{d}^{123} + \\ &+ (D^1\,\mathrm{d}x + E^1\,\mathrm{d}y - V\mathrm{d}p^1 - W\mathrm{d}q^1)\,\mathrm{d}^{23} + + \\ &+ (p_i\,\mathrm{d}p^i + q_i\,\mathrm{d}q^i)(C^1\,\mathrm{d}^{23} + +) \\ &- \mathrm{d}x[\mathrm{d}p^1(C^2\,\mathrm{d}^3 - C^3\,\mathrm{d}^2) + +] \\ &- \mathrm{d}y[\mathrm{d}q^1(C^2\,\mathrm{d}^3 - C^3\,\mathrm{d}^2) + +]. \end{aligned} \tag{8.6}$$

Finally

$$\begin{aligned} \mathrm{d}\zeta = &(\mathrm{d}\alpha - F^j\,\mathrm{d}p_j - G^j\,\mathrm{d}q_j - H_j\mathrm{d}t^j - I\,\mathrm{d}y)\,\mathrm{d}p_i\,\mathrm{d}t^i \\ &+ (\mathrm{d}\beta - J^j\,\mathrm{d}p_j - K^j\,\mathrm{d}q_j - L_j\,\mathrm{d}t^j - M\,\mathrm{d}x)\,\mathrm{d}q_i\,\mathrm{d}t^i \\ &+ (\mathrm{d}x - p_i\,\mathrm{d}t^i)(-\mathrm{d}F^j\,\mathrm{d}p_j - \mathrm{d}G^j\,\mathrm{d}q_j - \mathrm{d}H_j\,\mathrm{d}t^j - \mathrm{d}I\,\mathrm{d}y) \\ &+ (\mathrm{d}y - q_i\,\mathrm{d}t^i)(-\mathrm{d}J^j\,\mathrm{d}p_j - \mathrm{d}K^j\,\mathrm{d}q_j - \mathrm{d}L_j\,\mathrm{d}t^J - \mathrm{d}M\,\mathrm{d}x). \end{aligned} \tag{8.7}$$

The sum of (8,5, 8,6, 8,7) is presumably 0. It will therefore still be 0 if we replace $\mathrm{d}x$ by $p_i\,\mathrm{d}t^i$ and $\mathrm{d}y$ by $q_i\,\mathrm{d}t^i$. This leaves (8.5) alone, but turns (8.6) into

$$\begin{aligned} &-\{(V\,\mathrm{d}p^1 + W\,\mathrm{d}q^1)\,\mathrm{d}^{23} + +\} \\ &+ (p_i\,\mathrm{d}p^i + q_i\,\mathrm{d}q^i)(C^1\,\mathrm{d}^{23} + +) \end{aligned} \tag{8.6*}$$

$$-p_i\,\mathrm{d}t^i[\mathrm{d}p^1(C^2\,\mathrm{d}^3 - C^3\,\mathrm{d}^2) + \mathrm{d}p^2(C^3\,\mathrm{d}^1 - C^1\,\mathrm{d}^3) + \mathrm{d}p^3(C^1\,\mathrm{d}^2 + C^2\,\mathrm{d}^1)]$$
$$-q_i\,\mathrm{d}t^i[\mathrm{d}q^1(C^2\,\mathrm{d}^3 - C^3\,\mathrm{d}^2) + \mathrm{d}q^2(C^3\,\mathrm{d}^1 - C^1\,\mathrm{d}^3) + \mathrm{d}q^3(C^1\,\mathrm{d}^2 - C^2\,\mathrm{d}^1)].$$

It helps most for (8.7), yielding

$$\begin{aligned} &(\mathrm{d}\alpha - F^j\,\mathrm{d}p_j - G^j\,\mathrm{d}q_j - H_j\,\mathrm{d}t^j - I\mathrm{d}y)\,\mathrm{d}p_i\,\mathrm{d}t^i \\ &+ (\mathrm{d}\beta - J^j\,\mathrm{d}p_j - K^j\,\mathrm{d}q_j - L_j\,\mathrm{d}t^j - M\mathrm{d}x)\,\mathrm{d}q_i\,\mathrm{d}t^i \end{aligned} \tag{8.7*}$$

*where* $\mathrm{d}x = p_i\,\mathrm{d}t^i$ *and* $\mathrm{d}y = q_i\,\mathrm{d}t^i$.

We observe some relations:

$$\frac{\partial \alpha}{\partial p_i} - F^i = 0, \tag{8.7.1}$$

$$A^3 p^1 - A^1 p^3 + \frac{\partial \alpha}{\partial t^1} + p_1 \frac{\partial \alpha}{\partial x} - H_1 - I q_1 = 0. \tag{8.7.2}$$

The first of these, for $i = 1$, is the coefficient of $dp_1\, dp_2\, dt^2$ in (8.7*), and since (8.5, 8.6*) have no such terms, we get (8.7.1). The second relation comes from the coefficient of $dp_1\, dt^1\, dt^2$. There is actually some internal cancelling in (8.7*).

Now we return to the sum of (8.5, 8.6, 8.7) and look for the $dx\, dp^1\, dt^i$ terms. Their sum is

$$\left[-C^2 dt^3 + C^3 dt^2 + \frac{\partial \alpha}{\partial x} g_{11} dt^1 - \frac{\partial F_1}{\partial t^i} dt^i - \frac{\partial H_j}{\partial p^1} dt^j \right.$$
$$\left. + (p_i\, dt^i) \frac{\partial F_1}{\partial x} - q_i\, dt^i - \left(\frac{\partial J_1}{\partial x} + \frac{\partial M}{\partial p^1}\right)\right] dx\, dp^1.$$

The coefficient of $dx\, dp^1\, dt^1$ is set equal to 0:

$$\frac{\partial \alpha}{\partial x} g_{11} - \frac{\partial F_1}{\partial t^1} - \frac{\partial H_1}{\partial p^1} + p_1 \frac{\partial F_1}{\partial x} + q_1 \left(\frac{\partial J_1}{\partial x} - \frac{\partial M}{\partial p^1}\right) = 0. \tag{8.7.3}$$

From the coefficient of $dx\, dp^1\, dt^2$ we obtain

$$C^3 - \frac{\partial F_1}{\partial t^2} - \frac{\partial H_2}{\partial p^1} + p_2 \frac{\partial F_1}{\partial x} + q_2 \left(\frac{\partial J_1}{\partial x} - \frac{\partial M}{\partial p^1}\right) = 0.$$

Let us interchange 1 and 2 here. We obtain

$$C^3 - \frac{\partial F_2}{\partial t^1} - \frac{\partial H_1}{\partial p^2} + p_1 \frac{\partial F_2}{\partial x} + q_1 \left(\frac{\partial J_2}{\partial x} - \frac{\partial M}{\partial p^2}\right) = 0. \tag{8.7.4}$$

Now we solve (8.7.2) for $H^1$ and substitute into (8.7.3). After two applications of (8.7.1), this changes (8.7.3) to

$$q_1 \left(\frac{\partial J_1}{\partial x} - \frac{\partial M}{\partial p^1}\right) + q_1 \frac{\partial I}{\partial p^1} = 0.$$

Thus

$$(8.7.5) \qquad \frac{\partial J_1}{\partial x} - \frac{\partial M}{\partial p^1} + \frac{\partial I}{\partial p^1} = 0.$$

Changing 1 to 2 in (8.6.5) and using it here gives $C^3 = -A^3$. Of course this means $C^i = -A^i$. Interchanging $x$ and $y$, $p$ and $q$ we obtain also $C^k = -B^i$. The establishes (7.7).

Conversely, when $A = B$, then (7.1) is remotely Hamiltonic. This follows from (10.1) below.

## 9. THE ONE-DIMENSIONAL KLEIN GORDON FIELD

In contrast to the two-dimensional field just discussed, in the one-dimensional case, all dynamic forms are remotely dynamic. Before stating that result, we enumerate the various dynamic forms. We allow a somewhat more general field, namely that defined by

$$(9.1) \qquad H = \frac{1}{2} p_i p^i + h(x)$$

where $h$ is any analytic function.

It will be convenient to define a 3-form to be *dynamically null* if it is the sum

$$(9.2) \qquad \beta + \gamma$$

where $\beta$ is exact and $\gamma = (\mathrm{d}x - p_i\, \mathrm{d}x^i)\, \Theta$ where $\Theta$ is a 2-form.

9.3. THEOREM. *Consider the four cases*

$$(9.3.1) \qquad h(x) = a + bx + cx^2, \qquad c \neq 0$$

$$(9.3.2) \qquad h(x) = a + bx$$

$$(9.3.3) \qquad h(x) = a + (bx + c)^4, \qquad b \neq 0,$$

*where a, b, c are constants, and*

$$(9.3.4) \qquad h \quad \textit{is not as in } (9.3.1, 9.3.2, 9.3.3).$$

*The most general dynamic current for* (9.1) *is*

$$(9.3.5) \qquad J^1\, \mathrm{d}^{234} - J^2\, \mathrm{d}^{134} + J^3\, \mathrm{d}^{124} - J^4\, \mathrm{d}^{123} + \epsilon$$

*where $\epsilon$ is a dynamical null-form, and*

$$(9.3.6) \qquad J^i = \frac{1}{2} A^i p_j p^j - p^i A_j p^j + up^i - h(x) A^i - \frac{x^2}{2} a^i + \eta^i$$

(9.3.7) $$A^i = \frac{1}{2} a^i t_j t^j - t^i a_j t^j + \mu^{ij} t_j + \lambda t^i + b^i$$

*where these coefficients are constants, with* $\mu^{ij} = -\mu^{ji}$, *and* $u = \rho - (\lambda - a_i t^i)x$.

The restrictions on these constants and on the functions $\eta^i$ of $t$ and $x$ depend on the case, as follows.

*Case 9.3.1.* $A^i = \mu^{ij} t_j + b^i$, $\eta^i = -\dfrac{b + 2cx}{2c} \rho^i$, where

$$\rho^i_i + 2c\rho = 0, \qquad \lambda = a^j = 0.$$

Here $\rho^i$ means $\dfrac{\partial \rho}{\partial t_i}$ and $\rho^i_j = \dfrac{\partial \rho^i}{\partial t^j} = g^{ik} \dfrac{\partial^2 \rho}{\partial t^j \, \partial t^k}$.

*Case 9.3.2.* No restriction on the coefficients of $A^i$, but

$$\eta^i = -\rho^i x + 4a\left[\lambda t^i - \frac{1}{2} a^i (t^i)^2\right],$$

(no sum on $i$ intended); and $\rho^i_i + 3(\lambda - a_i t^i)\, b = 0$.

*Case 9.3.3.* No restrictions on $A^i$, and

$$\eta^i = -\frac{a^i c x}{b} + 4a\left[\lambda t^i - \frac{1}{2} a^i (t^i)^2\right], \quad \rho = -\frac{c(\lambda - a_i t^i)}{b}.$$

*Case 9.3.4.* $A^i = \mu^{ij} t_j + b^i$, and $\eta^i = 0$.

The proof is based on an analogue of (2.6), which we now present. We adjust our notation so that item (9.6.1) below corresponds to (2.6.1), and (9.6.4) corresponds to (2.6.4), etc..

9.6. THEOREM. *Let* $\delta = J^1 \mathrm{d}^{234} - J^2 \mathrm{d}^{134} + \ldots - \ldots$ *be a dynamic current. Then there exist functions* $A^1, \ldots, A^4$, $u$ *of* $t$ *and* $x$ *where*

(9.6.1) *The* $A^i$ *depend only* $t$ *and* $$\frac{\partial A_i}{\partial t^j} + \frac{\partial A_j}{\partial t^i} = 2g_{ij}\left(\frac{1}{2} \frac{\partial A^i}{\partial t^i} + \frac{\partial u}{\partial x}\right);$$

*there exist functions* $G^i$ *of* $t$ *and* $x$ *such that*

(9.6.4) $$\frac{\partial G^k}{\partial x} = -u^k - A^k h'(x),$$

(9.6.6) $$\frac{\partial G^k}{\partial t^k} = u h'(x);$$

*there exists a 3-form $\xi$ in the variables $t$ and $x$ as well as functions $C^{\lambda\mu}(= -C^{\mu\lambda}$, $\lambda, \mu = 1, 2, 3, 4, 5)$ of $t$ and $x$ such that*

(9.6.8) $$G^1 \mathrm{d}^{234} - G^2 \mathrm{d}^{134} + \ldots - \ldots + \mathrm{d}\xi = \Sigma\, \epsilon_{\lambda\mu\rho\sigma\tau} C^{\lambda\mu}\, \mathrm{d}^{\rho\sigma\tau}$$

*and*

(9.6.9) $$J^i = \frac{1}{2} A^i p_j p^j - p^i A_j p^j + u p^i + C^{i\mu} p_\mu$$

*where $p_5 = -1$.*

The proof is so much like that of (2.6) that it may be omitted. In fact, the proof is much shorter because it ends when we get to a place corresponding to (5.4).

The statement of (9.6) really corresponds to the converse of (2.6) since it describes the most general dynamic current. But the converse of (9.6) is also true.

To get from (9.6) to (9.3) we insert a lemma.

9.7. LEMMA. *Conditions* (9.6.1, 9.6.4, 9.6.6) *imply*

(9.7.1) $A^k$ *has the form* (9.3.7),

(9.7.2) $$u = \rho - \sigma x$$

*where*

(9.7.3) $$\sigma = \lambda - a_i t^i$$

*and*

(9.7.4) $\rho$ *depends only on* $t$.

(9.7.5) $$\frac{\partial A^k}{\partial t^k} = 4\sigma$$

(9.7.6) $$G^1 \mathrm{d}^{234} - G^2 \mathrm{d}^{134} + \ldots - \ldots$$

$$= - h(x)(A^1 \, d^{234} - \ldots) - \frac{x^2}{2} (a^1 \, d^{234} - \ldots) + \eta$$

*where*

(9.7.7) $\quad d\eta = (\rho^1 \, d^{234} - \ldots) \, dx + [4\sigma h + (\rho - \sigma x) \, h'] \, d^{1234}.$

*Here* $\rho^i$ *means* $\partial \rho / \partial t_i$.

An immediate consequence of (9.7.7) is

(9.7.8) $\quad \rho_i^i + 3\sigma h' + (\rho - \sigma x) \, h'' = 0.$

We give the proof of (9.7). (9.7.1) comes from (5.1). We now use (4.7.1) to calculate $\partial A^k / \partial t^k$, put it into (9.6.1) and obtain $\partial u / \partial x = - \lambda + a_i t^i$ which we call $- \sigma$. So $u = \rho - \sigma x$. We define

(9.7.9) $\quad \eta = G^1 \, d^{234} - \ldots + h(x)(A^1 \, d^{234} - \ldots) + \dfrac{x^2}{2} (a^1 \, d^{234} - \ldots)$

and calculate $d\eta$. Using (9.6.4, 9.6.5) and (9.7.7) we obtain (9.7.7). Finally, $dd\eta = 0$ gives (9.7.8).

It may be verified that the $a$, $\lambda$, $G^i$ etc. are uniquely determined by the given dynamic form except for the $\eta$. The dynamic form is represented only up to a dynamically null addendum.

We will take $\xi = 0$ in (9.6.8). Then

9.8. *Formula* (9.4.2) *will hold with the* $\eta^i$ *being defined by* $\eta = \eta^1 \, d^{234} - \eta^2 \, d^{134} + + \eta^3 \, d^{124} - \eta^4 \, d^{123}$.

In fact, if $\xi = 0$ then $G^1 \, d^{234}$ must be $\epsilon_{15234} C^{15} \, d^{234}$. The $\epsilon$-symbol is $-1$ so $C^{15} = - G^1$ and the sum $C^{1\mu} p_\mu$ in (9.6.9) reduces to $C^{15} p_5 = C^{15}(-1) = G^1$. We use (9.7.6) to express $G^1$ in terms of $\eta^1$ and this gives (9.4.2) (for $i = 1$, of course, but that will suffice).

REMARK. Theorem (9.3) shows that in the Klein-Gordon case (Case (9.3.1)) the vector field $A$ corresponding to a dynamic form has to be a Poincaré field. Hence more general conformal $A$ do not give rise to (or originate in) Hamiltonic forms. They are, nevertheless, symmetries of the Klein-Gordon field [3, p. 72, (4)].

Now we prove (9.3), starting with case (9.3.1). A result of (9.7.8) is $3\sigma \, 2c - - \sigma \, 2c = 0$ whence $\sigma = 0$. Thus $\lambda$ and the $a_j$ are 0, and $\rho_i^i + \rho h'' = 0$. Let

$$\eta = -\frac{2cx+b}{2c}(\rho^1 \mathrm{d}^{234} - \rho^2 \mathrm{d}^{124} + \ldots).$$

Then

$$\mathrm{d}\eta = -\frac{2cx+b}{2c}\rho^i_i \mathrm{d}^{1234} + (\rho^1 \mathrm{d}^{234} - \ldots)\,\mathrm{d}x$$

so that (9.7.7) is fulfilled. This leads to the $\eta^i$ listed.

For case (9.3.2), (9.7.8) says $\rho^i_i + 3b\sigma = 0$. (9.7.8) is the condition that $\eta$ can be found and we have presented a rather symmetrical choice.

In case (9.3.3), (9.7.8) takes the form

$$\rho^i_i + (bx+c)^2 12b(c\sigma + b\rho) = 0.$$

Now $b \neq 0$ so $c\sigma + b\rho = 0$ and $\rho^i_i = 0$. In fact, the former implies the latter and makes $\rho^i = ca^i/b$. It is easy to verify that

$$\rho = -\frac{cx}{b}(a^1 \mathrm{d}^{234} - \ldots) + 4a\left\{\left[\lambda t^1 - \frac{a_1}{2}(t^1)^2\right]\mathrm{d}^{234} - \ldots\right\}$$

gives a solution to (9.7.7).

In the last case, (9.3.4), we have $\rho^i_i + 3\sigma h' + (p - \sigma x)h'' = 0$. Let $\rho^i_i = \varphi$. Then $\varphi^i_i + \varphi h'' = 0$. If $\varphi$ were ever non-zero, then $h$ would be quadratic which was excluded. So $\rho^i_i = 0$, and $3\sigma h' + (\rho - \sigma x)h'' = 0$. If for any $x$ and $t$, $\rho - \sigma x$ were not 0, the $h'$ would be a constant multiple of $(\rho - \sigma x)^3$ and this was also excluded. Thus $\eta = 0$ satisfies (9.7.7).

## 10. RELATION TO HAMILTONIC FORMS

With the 1-dimensional Klein Gordon field, (in contrast to 2-dimensional case, (7.7)) each dynamic current is remotely Hamiltonic.

10.1. THEOREM. *Let $\delta$ be as in (9.6). Consider the vector field*

$$U = u\frac{\partial}{\partial x} + A^i\frac{\partial}{\partial t^i} + D^i\frac{\partial}{\partial p^i}$$

*where $A^i$ and $u$ are taken from (9.6.9) and*

$$D^i = -\frac{\partial u}{\partial x}p^i - \frac{\partial C^{i\mu}}{\partial x}p_\mu - h'(x)A^i + \epsilon^{ijk\ell}\frac{\partial G_{jk}}{\partial t^\ell} \tag{10.2}$$

*(no sum on i).*

*Let*

$$\zeta = (\mathrm{d}x - p_k\,\mathrm{d}t^k)\,G_{ij}\,\mathrm{d}t^i\,\mathrm{d}t^j$$

*where*

$$-G_{ji} = G_{ij}\,,\quad G_{12} = \frac{1}{2}(A^3p^4 - A^4p^3 + C^{34}),$$

$$G_{13} = -\frac{1}{2}(A^2p^4 - A^4p^2 + C^{24}),\quad \ldots$$

*the sign being that of the permutation of the four indices. Then*

$$\mathrm{d}\delta + U \lrcorner \mathrm{d}\alpha + \mathrm{d}\zeta = 0. \tag{10.3}$$

The proof is elementary and not too long if performed as follows. First we write down $\mathrm{d}\delta$ and $\mathrm{d}\zeta$. Then we note that

$$\begin{aligned} U \lrcorner \mathrm{d}\alpha = {} & (-\,uh' - p_jD^j)\,\mathrm{d}^{1234} \\ & + (-\,u\mathrm{d}p^1 + p_j\mathrm{d}p^jA^1)\mathrm{d}^{234} - (-u\mathrm{d}p^2 + p_j\,\mathrm{d}p^jA^2)\mathrm{d}^{134} + \ldots \\ & + \mathrm{d}x\,[(h'A^1 + D^1)\,\mathrm{d}^{234} - \ldots] + \\ & + \mathrm{d}x\,[-\,\mathrm{d}p^1(A^2\mathrm{d}^{34} - A^3\mathrm{d}^{24} + A^4\mathrm{d}^{23}) \\ & \qquad + \mathrm{d}p^2(A^1\mathrm{d}^{34} - A^3\mathrm{d}^{14} + A^4\mathrm{d}^{13}) \\ & \qquad - \mathrm{d}p^3(A^1\mathrm{d}^{24} - A^2\mathrm{d}^{14} + A^4\mathrm{d}^{12}) \\ & \qquad + \mathrm{d}p^4(A^1\mathrm{d}^{23} - A^2\mathrm{d}^{13} + A^3\mathrm{d}^{12})]. \end{aligned}$$

One hunts through the three expressions for the $\mathrm{d}p_1\,\mathrm{d}^{234}$ terms. The sum is zero. The same is true for the $\mathrm{d}p_1\,\mathrm{d}^{134}$ terms. Neither of these require the $D^i$ in (10.2). Thus (10.3) holds as far as $\mathrm{d}p\ \mathrm{d}t\ \mathrm{d}t\ \mathrm{d}t$ terms are concerned.

Then we look at the $\mathrm{d}x\ \mathrm{d}p_1\,\mathrm{d}^{ij}$ terms. Typically, the question is something like

$$g^{11}A^2 + 2\,\frac{\partial G_{34}}{\partial p_1} = 0\ ?$$

All of these can be verified.

Next we look at $\mathrm{d}x\ \mathrm{d}^{234}$. This leads to an equation which is essentially (10.2). Finally, we examine the $\mathrm{d}^{1234}$ term. Here we need (10.2), and the equation to be verified is precisely the 1-dimensional analogue of (2.7.3) with $\mu^1 = u - p_iA^i$.

This latter is the case by (5.1.3).

Thus ends our proof of (10.1). In spite of the complexity of (10.2), when $\zeta$ is chosen as 0, the vector field $U$ is a natural associate of the vector field

$$\varkappa = u \frac{\partial}{\partial x} + A^i \frac{\partial}{\partial t^i}$$

in $\mathbb{R}^4 \times \mathbb{R}$. As explained in [7], $\varkappa$ induces, or can be lifted up to, a vector field $\varkappa\uparrow$ in the bundle $J^1(\mathbb{R}^4, \mathbb{R})$. Our $U$ is nothing but this lift of $\varkappa$. We will give the proof elsewhere.

Could it be that $\delta$, being dynamic, is automatically Hamiltonic? The calculations used in verifying (10.1) can be adapted to show that this is not so.

10.4. PROPOSITION. *A dynamic $\delta$ need not be Hamiltonic.*

To prove this first «decouple» the $U$ from the $\delta$ by putting bars over the components of $U$. Suppose $\mathrm{d}\delta + U \lrcorner \mathrm{d}\alpha = 0$. The $\mathrm{d}p_1 \mathrm{d}^{134}$ equation which we have asked the reader to construct now reduces to $-J^{21} - p^1 \bar{A}^2 = 0$. If this is true then surely also $-J^{23} - p^3 \bar{A}^2 = 0$ and therefore $p^3 J^{12} = p^1 J^{23}$. This relation implies $A^1 = A^3 = 0$ and so of course also $A^2 = A^4 = 0$. This need hardly be. The 1-dimensional analogue of (7.1) is an example.

## 11. DYNAMIC FORMS WHICH ARE NOT CURRENTS

For the real 1-dimensional Klein-Gordon field we can say the following

11.1. THEOREM. *Let $\delta$ be a dynamic form. Then $\delta$ differs from a dynamic current by a dynamically null form.*

11.2. COROLLARY. *Let $\delta$ be a dynamic form. Then $\delta$ differs from a Hamiltonic form by a dynamically null form.*

*Sketch of proof of 11.1.* Let $\delta$ be any form. Replace $\mathrm{d}x$ by $p_i\,\mathrm{d}t^i$ and thus obtain $\delta'$. Then $\delta' - \delta$ is dynamically null. Thus we may suppose that $\delta$ has no $\mathrm{d}x$ terms. If it is not a current then there must be terms like

$$\mathrm{d}p_i\,\mathrm{d}t^j\,\mathrm{d}t^k, \quad \mathrm{d}p_i\,\mathrm{d}p_j\,\mathrm{d}t^k, \quad \text{or} \quad \mathrm{d}p_i\,\mathrm{d}p_j\,\mathrm{d}p_k.$$

The proof begins by showing that, if $\delta$ is dynamic and has terms $\mathrm{d}p\,\mathrm{d}p\,\mathrm{d}p$, then one can find a null-form $\mu$ such that $\delta + \mu$ has no such terms. This step is rather easy.

Let

$$\delta = L^1\, dp^2\, dp^3\, dp^4 - L^2\, dp^1\, dp^3\, dp^4 + L^3\, dp^1\, dp^2\, dp^4 - L^4\, dp^1\, dp^2\, dp^3 + M_2$$

where $M_2$ is the sum of the terms with at most two $dp$'s. Then

$$d\delta = \lambda\, dp^1 \ldots dp^4 + M_3,$$

where $\lambda = \partial L^i/\partial p^i$.

Now we form $\langle d\delta;\, U_1, U_2, U_3, U_4\rangle \equiv f$ with the intent of using (4.1). In fact

$$f = \lambda \epsilon_{ijk\ell}\, U_1^i\, U_2^j\, U_3^k\, U_4^\ell + f_3$$

where $f_3$ has terms at most of degree 3 in the $U_K^i$.

Let us replace $U_4^4$ by $-h'(x) - U_1^1 - U_2^2 - U_3^3$, and also replace $U_{ij}$ by $U_{ji}$ when $i > j$. Then $f$ should be 0. It is easy to see that

$$g = \lambda \epsilon_{ijk\ell}\, U_1^i\, U_2^j\, U_3^k\, U_4^\ell$$

should itself vanish when $U_{ij} = U_{ji}$ and $U_4^4 = -U_1^1 - U_2^2 - U_3^3$. We take

$$\frac{\partial^2 g}{(\partial U_1^1)^2} = -2\lambda \epsilon_{1jk4}\, U_2^j\, U_3^k.$$

This shows $\lambda = 0$. This means that $L^1\, dp^2\, dp^3\, dp^4 - \ldots$ is exact when the $t$'s and $x$ are constant. So there is a 3-form $\beta$ such that

$$d\beta = L^1\, dp^2\, dp^3\, dp^4 - \ \ldots + M_2$$

where $M_2$ has the same meaning as before. Thus $\delta' = \delta - d\beta$ or $\delta' - \delta$ is dynamically null.

Hence we should now turn to the case where $\delta$ has at most $dp\, dp\, dt$ terms. To save space we leave this case to the reader and turn to the case

$$\delta = N_{ijk}\, dp^i\, dt^j\, dt^k + M_0.$$

It gives an adequate idea of the technique (such as it is).

We can assume $N_{ijk} = -N_{ikj}$. Then

$$d\delta = \frac{\partial N_{ijk}}{\partial p^\ell}\, dp^\ell\, dp^i\, d^{jk} + M_1$$

from which

$$\langle d\delta, U_1, U_2, U_3, U_4\rangle \equiv f = \frac{\partial N_{ijk}}{\partial p^\ell}\, \epsilon^{abjk}\, U_a^\ell\, U_b^i + f_1$$

where $f_1$ means the terms of degree at most 1 in the $U$'s.

We replace $U_4^4$ by $-U_1^1 - U_2^2 - U_3^3$ and $U_{ij}$ by $U_{ji}$ for $i > j$ as before, and this surely makes

$$(11.3) \qquad g = \frac{\partial N_{ijk}}{\partial p^{\ell}} \epsilon^{abjk} U_a^{\ell} U_b^i$$

identically 0. Again, we examine the equation

$$\frac{\partial^2 g}{(\partial U_1^1)^2} = 0.$$

It says

$$\frac{\partial N_{423}}{\partial p^1} - \frac{\partial N_{123}}{\partial p^4} = 0.$$

This shows that for each $i$, $j$ with $i \neq j$ there exists an $N_{ij} (= -N_{ji})$ such that

$$(11.4) \qquad N_{kij} = \frac{\partial N_{ij}}{\partial p^k}$$

*whenever these* 3 *indices are distinct.*

Now we return to (11.3) and work out

$$\frac{\partial^2 g}{\partial U_1^1 \, \partial U_{12}} = 0.$$

Using

$$\frac{\partial U_a^{\ell}}{\partial U_{12}} = \delta_a^1 g^{\ell 2} + \delta_a^2 g^{\ell 1}$$

this equation reduces to

$$\frac{\partial N_{423}}{\partial p_2} + \frac{\partial N_{23}^2}{\partial p^4} + \frac{\partial N_{413}}{\partial p_1} - \frac{\partial N_{13}^1}{\partial p^4} = 0.$$

Here $N_{23}^2 = g^{2i} N_{i23}$ as usual, and $\dfrac{\partial N}{\partial p_2} = g^{2i} \dfrac{\partial N}{\partial p^i}$.

We can use (11.4) twice here, since, for example $N_{423} = \partial N_{23}/\partial p_4$ and obtain

$$\frac{\partial}{\partial p^4} \{\ldots\} = 0$$

where $\{\ldots\}$ is the left side of the next equation.

$$\frac{\partial N_{23}}{\partial p_2} + N^2_{23} + \frac{\partial N_{13}}{\partial p_1} - N^1_{13} = f(123) \tag{11.5}$$

where $f(123)$ means a function of $p_1, p_2, p_3$ (besided the $t$ and $x$).

Denote

$$\frac{\partial N_{13}}{\partial p_1} - N^1_{13} \quad \text{by} \quad A^{13}_1$$

and define $A^{ij}_i$ analogously. So

$$A^{13}_1 - A^{23}_2 = f(123).$$

Of course, also $A^{43}_4 - A^{13}_1 = f(413)$ and $A^{23}_2 - A^{43}_4 = f(243)$. Adding these three we get

$$0 = f(123) + f(413) + f(243).$$

Setting $p_4$ equal to some constant gives us

$$f(123) = f(23) - f(13).$$

Hence

$$A^{13}_1 - f(13) - [A^{23}_2 - f(23)] = 0.$$

Define $\bar{N}_{13} = N_{13} +$ any integral of $f(13)\,\mathrm{d}p_1$. Then (11.4) is preserved, in the sense that

$$\frac{\partial \bar{N}_{13}}{\partial p^k} = N_{k13} \quad \text{for} \quad k = 2, 4$$

which is to say (11.4) holds, and dropping the bars,

$$\frac{\partial N_{12}}{\partial p_1} - N^1_{13} = \frac{\partial N_{23}}{\partial p_2} - N^2_{23}.$$

Because of this equation, we may denote each of its sides by $f_3$, where this $f_3$ may depend on all the $p's$ as well as $t$ and $x$. So we have

$$N^i_{ij} = \frac{\partial N_{ij}}{\partial p_i} + f_j \quad \text{for} \quad j \neq i. \tag{11.6}$$

This relation supplements (11.4). The sum $N_{ijk}\,\mathrm{d}p^i\,\mathrm{d}^{jk}$ can be written as $N^i_{jk}\,\mathrm{d}p_i\,\mathrm{d}^{jk} = N^1_{jk}\,\mathrm{d}p_1\,\mathrm{d}^{jk} + N^2_{jk}\,\mathrm{d}p_2\,\mathrm{d}^{jk} + \ldots.$

We take a closer look at the $\mathrm{d}p_1$ part of this sum:

$$2\mathrm{d}p_1(N^1_{12}\,\mathrm{d}^{12} + N^1_{23}\,\mathrm{d}^{23} + N^1_{13}\,\mathrm{d}^{13} + N^1_{24}\,\mathrm{d}^{24} + N^1_{14}\,\mathrm{d}^{14} + N^1_{34}\,\mathrm{d}^{34})$$

and this is, by (11.6)

$$2\mathrm{d}p_1\left(\sum_{j<k} \frac{\partial N_{jk}}{\partial p_1}\,\mathrm{d}^{jk} + \sum_k f_k\,\mathrm{d}^{1k}\right).$$

Adding this to the $\mathrm{d}p_2$, $\mathrm{d}p_3$, $\mathrm{d}p_4$ terms we have

$$\delta = \frac{\partial N_{jk}}{\partial p_i}\,\mathrm{d}p_i\,\mathrm{d}^{jk} + 2(\mathrm{d}p_i\,\mathrm{d}t^i)\,f_k\,\mathrm{d}t^k + M_0.$$

Let

$$\beta = -2(\mathrm{d}p_i\,\mathrm{d}t^i)\,f_k\,\mathrm{d}t^k.$$

This vanishes on all extremals (see (8.2)). Let

$$\nu = -N_{jk}\,\mathrm{d}^{jk}.$$

Then $\delta + \beta + \mathrm{d}\gamma$ is of type $M_0$, and is thus a dynamic current. This ends our proof of (11.1).

REMARK. Elsewhere I have written down a complete proof of (11.1) for the case of 3-dimensional space time. I have no doubts that the step missing in the proof above, going from $M_2$ to $M_1$, can be constructed.

## 12. ACKNOWLEDGEMENT

I express my sincere gratitude for the hospitality shown me at C.N.R.S.-
-Luminy-Centre de Physique theorique, Marseille, especially by Professor D. Kastler.

## REFERENCES

[1] RICHARD ARENS, *Reducing the Order of a Lagrangean*, Pac. Jour. Math., **93** (1981) 1 - 11.

[2] RICHARD ARENS, *Manifestly Dynamic Forms in the Cartan Hamilton Treatment of Classical Fields*, Pac. Jour. Math., **93** (1981) 13 - 30.

[3] M. FLATO, J. SIMON and D. STERNHEIMER, *Conformal Covariance of Field Equations*, Ann. Physics, **61**, 78 - 97 (1970).

[4] HUBERT GOLDSCHMIDT and SHOLOMO STERNBERG, *The Hamilton-Cartan Formalism in the Calculus of Variations*, Ann. Inst. Fourier, Grenoble 23, (1973) 203 - 267.

[5] M. GOLUBITZSKY and V. GUILLEMIN, *Stable Mappings and their Singularities*, G.T.M. 14, Springer, 1973.

[6] J. KIJOWSKI, *A finite-dimensional Canonical Formalism in Classical Field Theory*, Commun. Mat. Phys., **30** (1973) 99 - 128.

[7] EMMY NOETHER, *The best reference is E. Bessel-Hagen, Uber die Erhaltungssätze der Elektrodynamik*, Math. Annalen **84**, 258 - 275, (1921).

*Manuscript received: April 8, 1983.*

*Paper presented by*
*M. Flato and A. Lichnerowicz*